

# The complexity of the $q$ -analog of the $n$ -cube

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## Abstract

We present a positive, combinatorial, good formula for the complexity (= number of spanning trees) of the  $q$ -analog of the  $n$ -cube. Our method also yields the explicit block diagonalization of the commutant of the  $GL(n, \mathbb{F}_q)$  action on  $B_q(n)$ , the set of all subspaces of an  $n$ -dimensional vector space over  $\mathbb{F}_q$ .

## 1 Introduction

This paper is a revised and expanded version of part of a previous paper by the author [S2]. We begin with a discussion of our motivating problem. The number of spanning trees of a graph  $G$  is called the *complexity* of  $G$  and denoted  $c(G)$ . The *hypercube*  $C(n)$  is the graph whose vertex set is the set  $B(n)$  of all subsets of the  $n$ -set  $[n] = \{1, 2, \dots, n\}$  and where two subsets  $X, Y \in B(n)$  are connected by an edge iff  $X \subseteq Y$  or  $Y \subseteq X$ , and  $||X| - |Y|| = 1$ . A classical result states that the eigenvalues of the Laplacian of  $C(n)$  are  $2k$ ,  $k = 0, \dots, n$ , with respective multiplicities  $\binom{n}{k}$ . It follows from the matrix tree theorem that

$$c(C(n)) = \frac{1}{2^n} \left\{ \prod_{k=1}^n (2k)^{\binom{n}{k}} \right\} = \prod_{k=2}^n (2k)^{\binom{n}{k}}. \quad (1)$$

We now define a  $q$ -analog of  $C(n)$ . Let  $q$  be a prime power and let  $B_q(n)$  denote the set of all subspaces of an  $n$ -dimensional vector space over the finite field  $\mathbb{F}_q$ . The  $q$ -binomial coefficient  $\begin{bmatrix} n \\ k \end{bmatrix}$  denotes the number of  $k$ -dimensional subspaces in  $B_q(n)$ . The  $q$ -analog  $C_q(n)$  of the hypercube is the graph whose vertex set is  $B_q(n)$ , and where subspaces  $X, Y \in B_q(n)$  are connected by an edge iff  $X \subseteq Y$  or  $Y \subseteq X$ , and  $|\dim(X) - \dim(Y)| = 1$ .

The problem of finding a formula for  $c(C_q(n))$  is significantly more involved than the classical (i.e.,  $q = 1$ ) case. The underlying reason seems to be that while the collection of all subsets forms an association scheme the collection of all subspaces does not. To the best of our knowledge the eigenvalues of the Laplacian of  $C_q(n)$  are not known. In this paper we present a positive, combinatorial, good formula for  $c(C_q(n))$ . This stops well short of actually finding the eigenvalues of the Laplacian but can be used to efficiently write down  $c(C_q(n))$  for any given  $n$ . Let us explain this. Consider an algorithm that, on input

$n$ , writes down the number  $c(C(n))$  as the output. Now this number is large, having exponentially many bits in its binary representation and therefore, to write it down explicitly will take time exponential in  $n$ . However, note that the binomial coefficients  $\binom{n}{k}$  have at most  $n$  bits in their binary representation and can be calculated in time polynomial in  $n$  using the Pascal triangle. Thus formula (1) above shows that, given  $n$ , we can write down the number  $c(C(n))$  in time polynomial in  $n$  using product and exponential notation. So we define a formula for  $c(C_q(n))$  to be *good* if, given  $n$ , we can write down  $c(C_q(n))$  in time polynomial in  $n$  using sum, product and exponential notation (treating  $q$  symbolically). The terms positive and combinatorial will be self explanatory after the statement of Theorems 1.1 and 3.2 below.

Let us first reformulate the original formula (1) for  $c(C(n))$  in order to bring out the similarity with our formula for  $c(C_q(n))$ . Note that the following reformulation is also a good formula for  $c(C(n))$ .

We have

$$\begin{aligned}
c(C(n)) &= \frac{1}{2^n} \left\{ \prod_{k=1}^n (2k)^{\binom{n}{k}} \right\} \\
&= \frac{1}{2^n} \left\{ \prod_{k=1}^n (2k) \right\} \left\{ \prod_{k=1}^{\lfloor n/2 \rfloor} \left( \prod_{j=k}^{n-k} (2j) \right)^{\binom{n}{k} - \binom{n}{k-1}} \right\} \\
&= n! \left\{ \prod_{k=1}^{\lfloor n/2 \rfloor} \left( \prod_{j=k}^{n-k} (2j) \right)^{\binom{n}{k} - \binom{n}{k-1}} \right\} \tag{2}
\end{aligned}$$

To see the equivalence of the first and second lines above note that, for  $1 \leq j \leq n/2$ , the exponent of  $2j$  in the numerator of the first line is  $\binom{n}{j}$  and in the numerator of the second line is also  $\binom{n}{j} = 1 + \binom{n}{1} - \binom{n}{0} + \dots + \binom{n}{j} - \binom{n}{j-1}$ . Since  $\binom{n}{k} = \binom{n}{n-k}$  the same conclusion holds for  $n/2 \leq j \leq n$ .

For  $k \in \mathbb{N} = \{0, 1, 2, \dots\}$  set  $[\mathbf{k}] = 1 + q + q^2 + \dots + q^{k-1}$ . Let  $k, n \in \mathbb{N}$  with  $k \leq n/2$ . For  $k \leq j \leq n - k + 1$ , define polynomials  $F_q(n, k, j)$  in  $q$  using the following recursion:

$$\begin{aligned}
F_q(n, k, n - k + 1) &= 1 \\
F_q(n, k, n - k) &= [\mathbf{k}] + [\mathbf{n} - \mathbf{k}]
\end{aligned}$$

and, for  $k \leq j < n - k$ ,

$$F_q(n, k, j) = ([\mathbf{j}] + [\mathbf{n} - \mathbf{j}])F_q(n, k, j + 1) - (q^k[\mathbf{j} + \mathbf{1} - \mathbf{k}][\mathbf{n} - \mathbf{k} - \mathbf{j}])F_q(n, k, j + 2). \tag{3}$$

In Section 2 we prove the following formula for  $c(C_q(n))$ , which is similar to formula (2) above, except that the explicit term  $\prod_{j=k}^{n-k} (2j)$  is replaced by a recursive calculation.

**Theorem 1.1** *We have*

$$c(C_q(n)) = F_q(n, 0, 1) \left\{ \prod_{k=1}^{\lfloor n/2 \rfloor} F_q(n, k, k)^{[\mathbf{n}] - [\mathbf{k} - 1]} \right\},$$

where  $F_q(n, 0, 1) = [\mathbf{1}][\mathbf{2}] \cdots [\mathbf{n}]$ .

Given  $n$ , the polynomials  $F_q(n, k, j)$  and  $[\mathbf{n}] - [\mathbf{k} - 1]$  can be efficiently calculated (in time polynomial in  $n$ ), using the recurrence (3) and the  $q$ -Pascal triangle respectively, and the formula above for  $c(C_q(n))$  is

clearly good in the technical sense we have defined. The following table, computed using Maple, gives the first five values of  $c(C_q(n))$ .

$$\begin{aligned}
c(C_q(1)) &= 1 \\
c(C_q(2)) &= [2]2^q \\
c(C_q(3)) &= [2][3](4 + 3q + q^2)^{q(1+q)} \\
c(C_q(4)) &= [2][3][4](8 + 12q + 12q^2 + 10q^3 + 4q^4 + 2q^5)^{q(1+q+q^2)} \\
&\quad \times (2 + 2q)^{q^2(q^2+1)} \\
c(C_q(5)) &= [2][3][4][5]F_q(5, 1, 1)^{q(1+q)(1+q^2)} \\
&\quad \times F_q(5, 2, 2)^{q^2(1+q+q^2+q^3+q^4)}
\end{aligned}$$

where  $F_q(5, 2, 2) = 4 + 8q + 7q^2 + 4q^3 + q^4$  and

$$F_q(5, 1, 1) = 16 + 36q + 53q^2 + 65q^3 + 69q^4 + 58q^5 + 42q^6 + 26q^7 + 13q^8 + 5q^9 + q^{10}.$$

The table above suggests that the formula for  $c(C_q(n))$  in Theorem 1.1 is positive, i.e., for  $0 \leq k \leq n/2$ , both  $\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix}$  and  $F_q(n, k, k)$  have nonnegative coefficients (as polynomials in  $q$ ). A special case of a result of Butler [B] shows that indeed  $\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix}$ ,  $k \leq n/2$  has nonnegative coefficients. In Theorem 3.2 of Section 3 we show that the polynomials  $F_q(n, k, j)$  also have nonnegative coefficients by giving an explicit positive combinatorial formula for them.

In [S1] we studied the singular values of the up operator on the Boolean algebra and used this to explicitly block diagonalize the commutant of the symmetric group action on  $B(n)$ . Theorem 2.1 of Section 2 gives the singular values of the up operator on the  $q$ -analog of the Boolean algebra and we then use this result to prove Theorem 1.1, which in essence is a block diagonalization of the Laplacian of  $C_q(n)$ . The Laplacian lies in the commutant of the  $GL(n, \mathbb{F}_q)$  action on  $B_q(n)$  and in Theorem 4.1 of Section 4 we generalize Theorem 1.1 by explicitly block diagonalizing this commutant.

## 2 Singular values

A (finite) *graded poset* is a (finite) poset  $P$  together with a *rank function*  $r : P \rightarrow \mathbb{N}$  such that if  $p'$  covers  $p$  in  $P$  then  $r(p') = r(p) + 1$ . The *rank* of  $P$  is  $r(P) = \max\{r(p) : p \in P\}$  and, for  $i = 0, 1, \dots, r(P)$ ,  $P_i$  denotes the set of elements of  $P$  of rank  $i$ .

For a finite set  $S$ , let  $V(S)$  denote the complex vector space with  $S$  as basis. Let  $P$  be a graded poset with  $n = r(P)$ . Then we have  $V(P) = V(P_0) \oplus V(P_1) \oplus \dots \oplus V(P_n)$  (vector space direct sum). An element  $v \in V(P)$  is *homogeneous* if  $v \in V(P_i)$  for some  $i$ , and if  $v \neq 0$ , we extend the notion of rank to nonzero homogeneous elements by writing  $r(v) = i$ . The *up operator*  $U : V(P) \rightarrow V(P)$  is defined, for  $p \in P$ , by  $U(p) = \sum_{p'} p'$ , where the sum is over all  $p'$  covering  $p$ . A *symmetric Jordan chain* (SJC) in  $V(P)$  is a sequence

$$s = (v_k, \dots, v_{n-k}), \quad k \leq n/2, \quad (4)$$

of nonzero homogeneous elements of  $V(P)$  such that  $r(v_i) = i$  for  $i = k, \dots, n-k$ ,  $U(v_{i-1}) = v_i$ , for  $i = k+1, \dots, n-k$ , and  $U(v_{n-k}) = 0$  (note that the elements of this sequence are linearly independent,

being nonzero and of different ranks). We say that  $s$  starts at rank  $k$  and ends at rank  $n - k$ . A *symmetric Jordan basis* (SJB) of  $V(P)$  is a basis of  $V(P)$  consisting of a disjoint union of symmetric Jordan chains in  $V(P)$ .

Let  $\langle, \rangle$  denote the standard inner product on  $V(P)$ , i.e.,  $\langle p, p' \rangle = \delta(p, p')$  (Kronecker delta) for  $p, p' \in P$ . The *length*  $\sqrt{\langle v, v \rangle}$  of  $v \in V(P)$  is denoted  $\|v\|$ .

Suppose we have an orthogonal SJB  $J(n)$  of  $V(P)$ . Normalize the vectors in  $J(n)$  to get an orthonormal basis  $J'(n)$ . Let  $(v_k, \dots, v_{n-k})$  be a SJC in  $J(n)$ . Put  $v'_u = \frac{v_u}{\|v_u\|}$  and  $\alpha_u = \frac{\|v_{u+1}\|}{\|v_u\|}$ ,  $k \leq u \leq n - k$  (we set  $v'_{k-1} = v_{n-k+1} = 0$ ). We have, for  $k \leq u \leq n - k$ ,

$$U(v'_u) = \frac{U(v_u)}{\|v_u\|} = \frac{v_{u+1}}{\|v_u\|} = \alpha_u v'_{u+1}. \quad (5)$$

Thus the matrix of  $U$  wrt  $J'(n)$  is in block diagonal form, with a block corresponding to each (normalized) SJC in  $J(n)$ , and with the block corresponding to  $(v'_k, \dots, v'_{n-k})$  above being a lower triangular matrix with subdiagonal  $(\alpha_k, \dots, \alpha_{n-k-1})$  and 0's elsewhere.

The *down operator*  $D : V(P) \rightarrow V(P)$  is defined, for  $p \in P$ , by  $D(p) = \sum_{p'} p'$ , where the sum is over all  $p'$  covered by  $p$ . Note that the matrices, in the standard basis  $P$ , of  $U$  and  $D$  are real and transposes of each other. Since  $J'(n)$  is orthonormal with respect to the standard inner product, it follows that the matrices of  $U$  and  $D$ , in the basis  $J'(n)$ , must be adjoints of each other. Thus, for  $k - 1 \leq u \leq n - k - 1$ , we must have (using (5) and the previous paragraph),

$$D(v'_{u+1}) = \alpha_u v'_u. \quad (6)$$

In particular, the subspace spanned by  $\{v_k, \dots, v_{n-k}\}$  is closed under  $U$  and  $D$ . We use this observation and identities (5) and (6) above without explicit mention in a few places in Section 4.

Another useful observation is the following: take scalars  $\beta_0, \beta_1, \dots, \beta_n$  and define the operator  $\gamma : V(P) \rightarrow V(P)$  by  $\gamma(p) = \beta_{r(p)} p$ ,  $p \in P$ . Since each element of the SJC  $(v_k, \dots, v_{n-k})$  is homogeneous, it follows from the definition of  $\gamma$  that the subspace spanned by  $\{v_k, \dots, v_{n-k}\}$  is closed under  $U$ ,  $D$  and  $\gamma$ .

The *Boolean algebra* is the graded poset of rank  $n$  obtained by partially ordering  $B(n)$  by containment (with rank of a subset given by cardinality). The  *$q$ -analog of the Boolean algebra* is obtained by partially ordering  $B_q(n)$  by inclusion. This gives a graded poset of rank  $n$  with rank of a subspace given by dimension. The following result is the  $q$ -analog of a result about  $B(n)$  proved in [S1], which in turn was motivated by Schrijver's fundamental paper [S].

**Theorem 2.1** *There exists a SJB  $J(q, n)$  of  $V(B_q(n))$  such that*

- (i) *The elements of  $J(q, n)$  are orthogonal with respect to  $\langle, \rangle$  (the standard inner product).*
- (ii) (Singular Values) *Let  $0 \leq k \leq n/2$  and let  $(v_k, \dots, v_{n-k})$  be any SJC in  $J(q, n)$  starting at rank  $k$  and ending at rank  $n - k$ . Then we have, for  $k \leq u < n - k$ ,*

$$\frac{\|v_{u+1}\|}{\|v_u\|} = \sqrt{q^k [\mathbf{u} + \mathbf{1} - \mathbf{k}] [\mathbf{n} - \mathbf{k} - \mathbf{u}]} \quad (7)$$

**Proof** We shall put together several standard results.

- (i) The map  $U^{n-2k} : V(B_q(n)_k) \rightarrow V(B_q(n)_{n-k})$ ,  $0 \leq k \leq n/2$  is well known to be bijective. It follows, using a standard Jordan canonical form argument, that an SJB of  $V(B_q(n))$  exists.

(ii) Now we show existence of an orthogonal SJB. We use the action of the group  $GL(n, \mathbb{F}_q)$  on  $B_q(n)$ . As is easily seen the existence of an orthogonal SJB of  $V(B_q(n))$  (under the standard inner product) follows from facts (a)-(d) below by an application of Schur's lemma:

(a) Existence of some SJB of  $V(B_q(n))$ .

(b)  $U$  is  $GL(n, \mathbb{F}_q)$ -linear.

(c) For  $0 \leq k \leq n$ ,  $V(B_q(n)_k)$  is a multiplicity free  $GL(n, \mathbb{F}_q)$ -module (this is well known).

(d) For a finite group  $G$ , a  $G$ -invariant inner product on an irreducible  $G$ -module is unique upto scalars.

(iii) Now we prove part (ii) of the Theorem. Define an operator  $H : V(B_q(n)) \rightarrow V(B_q(n))$  by

$$H(X) = ([\mathbf{k}] - [\mathbf{n} - \mathbf{k}])X, \quad X \in B_q(n)_k, \quad 0 \leq k \leq n.$$

It is easy to check that  $[U, D] = UD - DU = H$ . To see this, fix  $X \in B_q(n)_k$ , and note that  $UD(X) = [\mathbf{k}]X + \sum_Y Y$ , where the sum is over all  $Y \in B_q(n)_k$  with  $\dim(X \cap Y) = k - 1$ . Similarly,  $DU(X) = [\mathbf{n} - \mathbf{k}]X + \sum_Y Y$ , where the sum is over all  $Y \in B_q(n)_k$  with  $\dim(X \cap Y) = k - 1$ . Subtracting we get  $[U, D] = H$ .

Let  $J(q, n)$  be an orthogonal SJB of  $V(B_q(n))$  and let  $(v_k, \dots, v_{n-k})$  be a SJC in  $J(q, n)$  starting at rank  $k$  and ending at rank  $n - k$ . Put  $v'_j = \frac{v_j}{\|v_j\|}$  and  $\alpha_j = \frac{\|v_{j+1}\|}{\|v_j\|}$ ,  $k \leq j \leq n - k$ . We have, from (5) and (6),

$$U(v'_j) = \alpha_j v'_{j+1}, \quad D(v'_{j+1}) = \alpha_j v'_j, \quad k \leq j < n - k.$$

We need to show that

$$\alpha_j^2 = q^k [\mathbf{j} + \mathbf{1} - \mathbf{k}] [\mathbf{n} - \mathbf{k} - \mathbf{j}], \quad k \leq j < n - k. \quad (8)$$

We show this by induction on  $j$ . We have  $DU = UD - H$ . Now  $DU(v'_k) = \alpha_k D(v'_{k+1}) = \alpha_k^2 v'_k$  and  $(UD - H)(v'_k) = ([\mathbf{n} - \mathbf{k}] - [\mathbf{k}])v'_k$  (since  $D(v'_k) = 0$ ). Hence  $\alpha_k^2 = [\mathbf{n} - \mathbf{k}] - [\mathbf{k}] = q^k [\mathbf{n} - \mathbf{2k}]$ . Thus (8) holds for  $j = k$ .

As in the previous paragraph  $DU(v'_j) = \alpha_j^2 v'_j$  and  $(UD - H)(v'_j) = (\alpha_{j-1}^2 + [\mathbf{n} - \mathbf{j}] - [\mathbf{j}])v'_j$ . By induction, we may assume  $\alpha_{j-1}^2 = q^k [\mathbf{j} - \mathbf{k}] [\mathbf{n} - \mathbf{k} - \mathbf{j} + \mathbf{1}]$ . Thus we see that  $\alpha_j^2$  is

$$\begin{aligned} &= q^k [\mathbf{j} - \mathbf{k}] [\mathbf{n} - \mathbf{k} - \mathbf{j} + \mathbf{1}] + [\mathbf{n} - \mathbf{j}] - [\mathbf{j}] \\ &= q^k \left\{ ([\mathbf{j} + \mathbf{1} - \mathbf{k}] - q^{j-k}) ([\mathbf{n} - \mathbf{k} - \mathbf{j}] + q^{n-k-j}) \right\} + [\mathbf{n} - \mathbf{j}] - [\mathbf{j}] \\ &= q^k \left\{ [\mathbf{j} + \mathbf{1} - \mathbf{k}] [\mathbf{n} - \mathbf{k} - \mathbf{j}] + q^{n-k-j} [\mathbf{j} + \mathbf{1} - \mathbf{k}] - q^{j-k} [\mathbf{n} - \mathbf{k} - \mathbf{j}] - q^{n-2k} \right\} \\ &\quad + [\mathbf{n} - \mathbf{j}] - [\mathbf{j}] \\ &= q^k [\mathbf{j} + \mathbf{1} - \mathbf{k}] [\mathbf{n} - \mathbf{k} - \mathbf{j}] + q^{n-j} [\mathbf{j} + \mathbf{1} - \mathbf{k}] - q^j [\mathbf{n} - \mathbf{k} - \mathbf{j}] - q^{n-k} \\ &\quad + [\mathbf{n} - \mathbf{j}] - [\mathbf{j}] \\ &= q^k [\mathbf{j} + \mathbf{1} - \mathbf{k}] [\mathbf{n} - \mathbf{k} - \mathbf{j}] + [\mathbf{n} + \mathbf{1} - \mathbf{k}] - [\mathbf{n} - \mathbf{j}] - [\mathbf{n} + \mathbf{1} - \mathbf{k}] + [\mathbf{j}] \\ &\quad + [\mathbf{n} - \mathbf{j}] - [\mathbf{j}] \\ &= q^k [\mathbf{j} + \mathbf{1} - \mathbf{k}] [\mathbf{n} - \mathbf{k} - \mathbf{j}], \end{aligned}$$

completing the proof.  $\square$

**Remark** The proof in [S1] of the  $q = 1$  case of Theorem 2.1 was constructive, giving a simple algorithm to explicitly write down an orthogonal SJB of  $V(B(n))$ . It would be interesting to give an explicit construction of an orthogonal SJB of  $V(B_q(n))$ .

For  $0 \leq k \leq n/2$ , define a real, symmetric, tridiagonal matrix  $N = N(k, n - k, n)$  of size  $n - 2k + 1$ , with rows and columns indexed by the set  $\{k, k + 1, \dots, n - k\}$ , and with entries given as follows: for  $k \leq i, j \leq n - k$  define

$$N(i, j) = \begin{cases} -\sqrt{q^k[\mathbf{j} - \mathbf{k}][\mathbf{n} - \mathbf{k} - \mathbf{j} + \mathbf{1}]} & \text{if } i = j - 1 \\ [\mathbf{j}] + [\mathbf{n} - \mathbf{j}] & \text{if } i = j \\ -\sqrt{q^k[\mathbf{j} + \mathbf{1} - \mathbf{k}][\mathbf{n} - \mathbf{k} - \mathbf{j}]} & \text{if } i = j + 1 \\ 0 & \text{if } |i - j| \geq 2 \end{cases}$$

For  $0 \leq k \leq n/2$  and  $k \leq j \leq n - k + 1$  define  $N_j = N_j(k, n - k, n)$  to be the principal submatrix of  $N = N(k, n - k, n)$  indexed by the rows and columns in the set  $\{j, j + 1, \dots, n - k\}$ . Thus,  $N_k = N$  and  $N_{n-k+1}$  is the empty matrix, which by convention has determinant 1.

**Lemma 2.2** For  $0 \leq k \leq n/2$  and  $k \leq j \leq n - k + 1$  we have

(i)  $F_q(n, k, j) = \det(N_j(k, n - k, n))$ .

(ii)  $F_q(n, 0, j) = [\mathbf{n}][\mathbf{n} - \mathbf{1}] \cdots [\mathbf{j}]$ .

In particular,  $\det(N_1(0, n, n)) = [\mathbf{n}][\mathbf{n} - \mathbf{1}] \cdots [\mathbf{1}]$ .

**Proof** (i) By (reverse) induction on  $j$ . The base cases  $j = n - k + 1, n - k$  are clear and the general case follows by expanding the determinant of  $N_j$  along its first column.

(ii) By (reverse) induction on  $j$ . The base cases  $j = n + 1, n$  are clear. By induction and the defining recurrence for  $F_q(n, k, j)$  we have

$$\begin{aligned} F_q(n, 0, j) &= ([\mathbf{j}] + [\mathbf{n} - \mathbf{j}])F_q(n, 0, j + 1) - ([\mathbf{j} + \mathbf{1}][\mathbf{n} - \mathbf{j}])F_q(n, 0, j + 2) \\ &= ([\mathbf{j}] + [\mathbf{n} - \mathbf{j}])[ \mathbf{n} ] \cdots [\mathbf{j} + \mathbf{1}] - ([\mathbf{j} + \mathbf{1}][\mathbf{n} - \mathbf{j}])[ \mathbf{n} ] \cdots [\mathbf{j} + \mathbf{2}] \\ &= [\mathbf{n}] \cdots [\mathbf{j}], \end{aligned}$$

completing the proof.  $\square$

We now prove our first formula for  $c(C_q(n))$ .

**Proof** (of Theorem 1.1)

The degree of a vertex  $X$  of  $C_q(n)$  is  $[\mathbf{k}] + [\mathbf{n} - \mathbf{k}]$ , where  $k = \dim(X)$ . Define an operator  $\deg : V(B_q(n)) \rightarrow V(B_q(n))$  by

$$\deg(X) = ([\mathbf{k}] + [\mathbf{n} - \mathbf{k}])X, \quad X \in B_q(n)_k.$$

We can now write the Laplacian  $L : V(B_q(n)) \rightarrow V(B_q(n))$  of  $C_q(n)$  as  $L = \deg - U - D$ , where  $U, D$  are the up and down operators on  $V(B_q(n))$ .

Let  $J(q, n)$  be a SJB of  $V(B_q(n))$  satisfying the conditions of Theorem 2.1. Normalize  $J(q, n)$  to get an orthonormal basis  $J'(q, n)$ . Since the vertex degrees are constant on  $B_q(n)_k$  it follows that the subspace spanned by each SJC in  $J(q, n)$  is closed under  $L$ . Using part (ii) of Theorem 2.1 we can write down the matrix of  $L$  in the basis  $J'(q, n)$ .

Let  $0 \leq k \leq n/2$ . Let  $(w_k, \dots, w_{n-k})$  be a SJC in  $J(q, n)$  starting at rank  $k$ . Set  $v_i = \frac{w_i}{\|w_i\|}$ ,  $k \leq i \leq n - k$ . Let  $W$  be the subspace spanned by  $\{v_k, \dots, v_{n-k}\}$ . Then  $W$  is invariant under  $L$ .

It follows from Theorem 2.1 that  $N(k, n - k, n)$  is the matrix of  $L : W \rightarrow W$  with respect to the (ordered) basis  $\{v_k, \dots, v_{n-k}\}$  (we take coordinate vectors with respect to a basis as column vectors). Thus the matrix of  $L$  with respect to (a suitable ordering of)  $J'(q, n)$  is in block diagonal form, with blocks  $N(k, n - k, n)$ , for all  $0 \leq k \leq n/2$ , and each such block is repeated  $\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix}$  times. The number of distinct blocks is  $1 + \lfloor n/2 \rfloor$ .

The unique element in  $J'(q, n)$  of rank 0 is the vector  $\mathbf{0}$  (here  $\mathbf{0}$  is the zero subspace).

Let  $\mathcal{M}$  denote the matrix of the Laplacian of  $C_q(n)$  in the standard basis  $B_q(n)$  and let  $\mathcal{M}'$  be obtained from  $\mathcal{M}$  by removing the row and column corresponding to vertex  $\mathbf{0}$ . From the matrix tree theorem we have  $c(C_q(n)) = \det(\mathcal{M}')$ . A little reflection shows that, by changing bases from  $B_q(n) - \{\mathbf{0}\}$  to  $J'(q, n) - \{\mathbf{0}\}$ ,  $\mathcal{M}'$  block diagonalizes with a block  $N_1(0, n, n)$  of multiplicity 1 and blocks  $N(k, n - k, n)$ ,  $1 \leq k \leq n/2$ , of multiplicity  $\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix}$ . The result now follows from Lemma 2.2.  $\square$

**Remark** Since the subdiagonal entries of  $N(k, n - k, n)$ ,  $0 \leq k \leq n/2$  are nonzero it easily follows that any eigenspace will have dimension 1 and thus  $N(k, n - k, n)$  has  $n - 2k + 1$  distinct eigenvalues. Data suggest that  $N(k, n - k, n)$  and  $N(l, n - l, n)$ ,  $k \neq l$  do not have any eigenvalue in common. In other words, the Laplacian of  $C_q(n)$  seems to have  $\sum_{k=0}^{\lfloor n/2 \rfloor} (n - 2k + 1) = (\lfloor n/2 \rfloor + 1)(\lfloor n/2 \rfloor + 1)$  distinct eigenvalues, with each of the  $n - 2k + 1$  eigenvalues of  $N(k, n - k, n)$  having multiplicity  $\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix}$ ,  $k = 0, 1, \dots, \lfloor n/2 \rfloor$ . Is this true and can it be proved without explicitly writing down the eigenvalues.

### 3 Positivity

In this section we define certain combinatorial objects and a generating function based on them with the property that an appropriate positive specialization satisfies the recurrence (3).

Let  $[\overline{n}] = \{\overline{1}, \overline{2}, \dots, \overline{n}\}$  and consider the set  $[n, \overline{n}] = [n] \cup [\overline{n}]$  of  $2n$  elements. We are going to recursively define a set  $S(n)$  of certain subsets of  $[n, \overline{n}]$ . The cardinality of an element of  $S(n)$  will be between 0 and  $n$  (inclusive) and will have the same parity as  $n$ . Define  $S(0) = \{\emptyset\}$ ,  $S(1) = \{\{1\}, \{\overline{1}\}\}$  and, for  $n \geq 1$ ,

$$S(n+1) = \{X \cup \{n+1\} : X \in S(n)\} \cup \{X \cup \{\overline{n+1}\} : X \in S(n), n \notin X\} \cup S(n-1).$$

It is easy to show by induction that  $|S(n)| = 2^n$  and that the number of elements of  $S(n)$  not containing  $n$  is  $2^{n-1}$ . Let  $\mathbf{x} = (x_1, x_2, \dots)$ ,  $\mathbf{y} = (y_1, y_2, \dots)$ , and  $z$  be indeterminates. For  $n \geq 0$  define the following polynomial

$$P(n, \mathbf{x}, \mathbf{y}, z) = \sum_{X \in S(n)} \left( \prod_{i \in X \cap [n]} x_i \right) \left( \prod_{\overline{i} \in X \cap [\overline{n}]} y_i \right) z^{\frac{n-|X|}{2}}.$$

**Example** We have

$$\begin{aligned} S(2) &= \{\emptyset, \{1, 2\}, \{\overline{1}, 2\}, \{\overline{1}, \overline{2}\}\}, \\ S(3) &= \{\{1\}, \{\overline{1}\}, \{3\}, \{\overline{3}\}, \{1, 2, 3\}, \{\overline{1}, 2, 3\}, \{\overline{1}, \overline{2}, 3\}, \{\overline{1}, \overline{2}, \overline{3}\}\}. \end{aligned}$$



Thus  $P(2, \mathbf{x}, \mathbf{y}, z) = z + (x_1x_2 + y_1x_2 + y_1y_2)$  and

$$P(3, \mathbf{x}, \mathbf{y}, z) = (x_1 + y_1 + x_3 + y_3)z + (x_1x_2x_3 + y_1x_2x_3 + y_1y_2x_3 + y_1y_2y_3).$$

The recursive structure of  $S(n)$  yields the following recurrence for the polynomials  $P$ .

**Theorem 3.1** *We have*

$$P(n+1, \mathbf{x}, \mathbf{y}, z) = (x_{n+1} + y_{n+1})P(n, \mathbf{x}, \mathbf{y}, z) - (x_n y_{n+1} - z)P(n-1, \mathbf{x}, \mathbf{y}, z), \quad n \geq 1.$$

**Proof** Let  $X \in S(n+1)$ . In the expansion of the lhs  $P(n+1, \mathbf{x}, \mathbf{y}, z)$ , consider the term corresponding to  $X$ :

$$\left( \prod_{i \in X \cap [n]} x_i \right) \left( \prod_{\bar{i} \in X \cap [\bar{n}]} y_{\bar{i}} \right) z^{\frac{n+1-|X|}{2}}.$$

We consider three cases:

- (i)  $n+1 \in X$ : the term above will appear exactly once in  $x_{n+1}P(n, \mathbf{x}, \mathbf{y}, z)$ .
- (ii)  $\overline{n+1} \in X$ : the term above will appear exactly once in  $y_{n+1}P(n, \mathbf{x}, \mathbf{y}, z) - x_n y_{n+1}P(n-1, \mathbf{x}, \mathbf{y}, z)$ .
- (iii)  $X \in S(n-1)$ : The term above will appear exactly once in  $zP(n-1, \mathbf{x}, \mathbf{y}, z)$ .

It is clear that there are no other terms corresponding to  $X$  on the rhs. The result follows.  $\square$

Given  $n, k \in \mathbb{N}$  with  $k \leq n/2$  define

$$d_q(n, k) = ([\mathbf{n} - \mathbf{k}], [\mathbf{n} - \mathbf{k} - \mathbf{1}], \dots, [\mathbf{k}], 0, 0, \dots), \quad e_q(n, k) = ([\mathbf{k}], [\mathbf{k} + \mathbf{1}], \dots, [\mathbf{n} - \mathbf{k}], 0, 0, \dots).$$

We now prove the nonnegativity of the coefficients of  $F_q(n, k, j)$ .

**Theorem 3.2** *Let  $n, k \in \mathbb{N}$  with  $k \leq n/2$  and let  $k \leq j \leq n - k + 1$ . Then*

$$F_q(n, k, j) = P(n - k - j + 1, d_q(n, k), e_q(n, k), [\mathbf{k}][\mathbf{n} - \mathbf{k} + \mathbf{1}]).$$

**Proof** The result is clearly true for  $j = n - k + 1$  and  $j = n - k$ . Now note the following alternate expression for the singular values:

$$\begin{aligned} q^k [\mathbf{j} + \mathbf{1} - \mathbf{k}][\mathbf{n} - \mathbf{k} - \mathbf{j}] &= ([\mathbf{j} + \mathbf{1}] - [\mathbf{k}])([\mathbf{n} - \mathbf{j}] - q^{n-k-j}[\mathbf{k}]) \\ &= [\mathbf{j} + \mathbf{1}][\mathbf{n} - \mathbf{j}] - [\mathbf{k}]( [\mathbf{n} - \mathbf{j}] + q^{n-k-j}[\mathbf{j} + \mathbf{1}] - q^{n-k-j}[\mathbf{k}] ) \\ &= [\mathbf{j} + \mathbf{1}][\mathbf{n} - \mathbf{j}] - [\mathbf{k}][\mathbf{n} - \mathbf{k} + \mathbf{1}] \end{aligned}$$

It now follows from Theorem 3.1 that  $P(n - k - j + 1, d_q(n, k), e_q(n, k), [\mathbf{k}][\mathbf{n} - \mathbf{k} + \mathbf{1}])$  satisfies the same recurrence as  $F_q(n, k, j)$ . The result follows.  $\square$

**Example** Let  $n = 3$  and  $k = 0$ . Then

$$d_q(3, 0) = ([\mathbf{3}], [\mathbf{2}], [\mathbf{1}], [\mathbf{0}], 0, 0, \dots), \quad e_q(3, 0) = ([\mathbf{0}], [\mathbf{1}], [\mathbf{2}], [\mathbf{3}], 0, 0, \dots).$$

Thus  $F_q(3, 0, 1) = P(3, d_q(3, 0), e_q(3, 0), 0) = [\mathbf{3}][\mathbf{2}][\mathbf{1}]$ .

Now let  $n = 3$  and  $k = 1$ . Then

$$d_q(3, 1) = ([\mathbf{2}], [\mathbf{1}], 0, 0, \dots), \quad e_q(3, 1) = ([\mathbf{1}], [\mathbf{2}], 0, 0, \dots).$$



Thus  $F_q(3, 1, 1) = P(2, d_q(3, 1), e_q(3, 1), [1][3]) = [1][3] + [2][1] + [1][1] + [1][2] = 4 + 3q + q^2$ , agreeing with the formula given in the introduction.

Taking  $d(n, k) = (n-k, n-k-1, \dots, k, 0, 0, \dots)$ ,  $e(n, k) = (k, k+1, \dots, n-k, 0, 0, \dots)$ , substituting  $q = 1$  in the formula above and comparing with (2) we get the following identity

$$P(n - 2k + 1, d(n, k), e(n, k), k(n - k + 1)) = 2^{n-2k+1} \cdot k \cdot (k + 1) \cdots (n - k).$$

## 4 Explicit block diagonalization

The group  $G = GL(n, \mathbb{F}_q)$  has a rank and order preserving action on the graded poset  $B_q(n)$ . Note that the Laplacian of  $C_q(n)$  belongs to the algebra  $\text{End}_G(V(B_q(n)))$ . In this section we generalize Theorem 1.1 by explicitly block diagonalizing  $\text{End}_G(V(B_q(n)))$ . Our method is the  $q$ -analog of the method used in [S1] to explicitly block diagonalize the commutant of the symmetric group action on  $B(n)$ .

We represent elements of  $\text{End}(V(B_q(n)))$  (in the standard basis) as  $B_q(n) \times B_q(n)$  matrices (we think of elements of  $V(B_q(n))$  as column vectors with coordinates indexed by  $B_q(n)$ ). For  $X, Y \in B_q(n)$ , the entry in row  $X$ , column  $Y$  of a matrix  $M$  will be denoted  $M(X, Y)$ . The matrix corresponding to  $f \in \text{End}(V(B_q(n)))$  is denoted  $M_f$ . Set  $\mathcal{A}(q, n) = \{M_f : f \in \text{End}_G(V(B_q(n)))\}$ . Then  $\mathcal{A}(q, n)$  is a  $*$ -algebra of matrices.

Let  $f \in \text{End}(V(B_q(n)))$  and  $g \in G$ . Then

$$f(g(Y)) = \sum_X M_f(X, g(Y))X \text{ and } g(f(Y)) = \sum_X M_f(X, Y)g(X).$$

It follows that  $f$  is  $G$ -linear if and only if  $M_f(X, Y) = M_f(g(X), g(Y))$ , for all  $X, Y \in B_q(n)$ ,  $g \in G$ , i.e.,  $M_f$  is constant on the orbits of the  $G$ -action on  $B_q(n) \times B_q(n)$ .

For  $0 \leq i, j, t \leq n$  let  $M_{i,j}^t$  be the  $B_q(n) \times B_q(n)$  matrix given by

$$M_{i,j}^t(X, Y) = \begin{cases} 1 & \text{if } \dim(X) = i, \dim(Y) = j, \dim(X \cap Y) = t \\ 0 & \text{otherwise} \end{cases}$$

Now  $(X, Y), (X', Y') \in B_q(n) \times B_q(n)$  are in the same  $G$ -orbit iff  $\dim(X) = \dim(X')$ ,  $\dim(Y) = \dim(Y')$ , and  $\dim(X \cap Y) = \dim(X' \cap Y')$ . It follows that

$$\{M_{i,j}^t \mid i - t + t + j - t \leq n, i - t, t, j - t \geq 0\}$$

is a basis of  $\mathcal{A}(q, n)$  and its cardinality is  $\binom{n+3}{3}$ .

Fix  $i, j \in \{0, \dots, n\}$ . Then we have

$$M_{i,t}^t M_{t,j}^t = \sum_{u=0}^n \begin{bmatrix} \mathbf{u} \\ \mathbf{t} \end{bmatrix} M_{i,j}^u, \quad t = 0, \dots, n,$$

since the entry of the lhs in row  $X$ , col  $Y$  with  $\dim(X) = i$ ,  $\dim(Y) = j$  is equal to the number of common subspaces of  $X$  and  $Y$  of size  $t$ . Apply  $q$ -binomial inversion (see Exercise 2.47 in [A]) to get

$$M_{i,j}^t = \sum_{u=0}^n (-1)^{u-t} q^{\binom{u-t}{2}} \begin{bmatrix} \mathbf{u} \\ \mathbf{t} \end{bmatrix} M_{i,u}^u M_{u,j}^u, \quad t = 0, \dots, n. \quad (9)$$

For the rest of this section set  $m = \lfloor n/2 \rfloor$ , and  $p_k = n - 2k + 1$ ,  $q_k = \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix}$ ,  $k = 0, \dots, m$ . Note that

$$\sum_{k=0}^m p_k^2 = \binom{n+3}{3}, \quad (10)$$

since both sides are polynomials in  $l$  (treating the cases  $n = 2l$  and  $n = 2l + 1$  separately) of degree 3 and agree for  $l = 0, 1, 2, 3$ .

Consider an orthogonal SJB  $J(q, n)$  of  $V(B_q(n))$  satisfying the conditions of Theorem 2.1. Normalize  $J(q, n)$  to get an orthonormal basis  $J'(q, n)$ . Let  $N(n)$  be the square  $B_q(n) \times J'(q, n)$  matrix, where, for  $v \in J'(q, n)$ , the column of  $N(n)$  indexed by  $v$  is the coordinate vector of  $v$  (in the standard coordinates  $B_q(n)$ ). By Theorem 2.1(i),  $N(n)$  is unitary. Since the action of  $M_{i,u}^u$  on  $V(B_q(n)_u)$  is  $\frac{1}{[i-u][i-u-1]\dots[1]}$  times the action of  $U^{i-u}$  on  $V(B_q(n)_u)$ , it follows by Theorem 2.1(ii) and identities (9), (10) above that  $N(n)^* \mathcal{A}(q, n) N(n)$  consists of all  $J'(q, n) \times J'(q, n)$  block diagonal matrices with a block corresponding to each (normalized) SJC in  $J(q, n)$  and any two SJC's starting and ending at the same rank giving rise to identical blocks. So there are  $q_k$  identical blocks of size  $p_k$ , for  $k = 0, \dots, m$ . It will be convenient to reindex the rows and columns of a block corresponding to a SJC starting at rank  $k$  and ending at rank  $n - k$  by the set  $\{k, k+1, \dots, n-k\}$ .

Define a map (below  $\text{Mat}(n \times n)$  denotes the algebra of complex  $n \times n$  matrices)

$$\Phi : \mathcal{A}(q, n) \cong \bigoplus_{k=0}^m \text{Mat}(p_k \times p_k),$$

by conjugating with  $N(n)$  followed by dropping duplicate blocks. We now write down the image  $\Phi(M_{i,j}^t)$ .

For  $i, j, k, t \in \{0, \dots, n\}$  define

$$\beta_{i,j,k}^{n,t}(q) = \sum_{u=0}^n (-1)^{u-t} q^{\binom{u-t}{2} - ku} \begin{bmatrix} u \\ t \end{bmatrix} \begin{bmatrix} n-2k \\ u-k \end{bmatrix} \begin{bmatrix} n-k-u \\ i-u \end{bmatrix} \begin{bmatrix} n-k-u \\ j-u \end{bmatrix}.$$

For  $0 \leq k \leq m$  and  $k \leq i, j \leq n-k$ , define  $E_{i,j,k}$  to be the  $p_k \times p_k$  matrix, with rows and columns indexed by  $\{k, k+1, \dots, n-k\}$ , and with entry in row  $i$  and column  $j$  equal to 1 and all other entries 0.

In the proof of the following result we will need another alternate expression for the singular values:

$$\sqrt{q^k [u+1-k][n-k-u]} = q^{\frac{k}{2}} [n-k-u] \begin{bmatrix} n-2k \\ u-k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-2k \\ u+1-k \end{bmatrix}^{-\frac{1}{2}}. \quad (11)$$

**Theorem 4.1** *Let  $i, j, t \in \{0, \dots, n\}$ . Write*

$$\Phi(M_{i,j}^t) = (N_0, \dots, N_m),$$

*where, for  $k = 0, \dots, m$ , the rows and columns of  $N_k$  are indexed by  $\{k, k+1, \dots, n-k\}$ . Then, for  $0 \leq k \leq m$ ,*

$$N_k = \begin{cases} q^{\frac{k(i+j)}{2}} \begin{bmatrix} n-2k \\ i-k \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} n-2k \\ j-k \end{bmatrix}^{-\frac{1}{2}} \beta_{i,j,k}^{n,t}(q) E_{i,j,k} & \text{if } k \leq i, j \leq n-k \\ 0 & \text{otherwise} \end{cases}$$

**Proof** Fix  $0 \leq k \leq m$ . If both  $i, j$  are not elements of  $\{k, \dots, n-k\}$  then clearly  $N_k = 0$ . So we may assume  $k \leq i, j \leq n-k$ . Clearly,  $N_k = \lambda E_{i,j,k}$  for some  $\lambda$ . We now find  $\lambda = N_k(i, j)$ .

Let  $u \in \{0, \dots, n\}$ . Write  $\Phi(M_{i,u}^u) = (A_0^u, \dots, A_m^u)$ . We claim that

$$A_k^u = \begin{cases} q^{\frac{k(i-u)}{2}} \begin{bmatrix} n-k-u \\ i-u \end{bmatrix} \begin{bmatrix} n-2k \\ u-k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-2k \\ i-k \end{bmatrix}^{-\frac{1}{2}} E_{i,u,k} & \text{if } k \leq u \leq n-k \\ 0 & \text{otherwise} \end{cases}$$

The otherwise part of the claim is clear. If  $k \leq u \leq n-k$  and  $i < u$  then we have  $A_k^u = 0$ . This also follows from the rhs since the  $q$ -binomial coefficient  $\begin{bmatrix} a \\ b \end{bmatrix}$  is 0 for  $b < 0$ . So we may assume that  $k \leq u \leq n-k$  and  $i \geq u$ . Clearly, in this case we have  $A_k^u = \alpha E_{i,u,k}$ , for some  $\alpha$ . We now determine  $\alpha = A_k^u(i, u)$ . We have using Theorem 2.1(ii) and the expression (11)

$$\begin{aligned} A_k^u(i, u) &= \frac{\prod_{w=u}^{i-1} \left\{ q^{\frac{k}{2}} \begin{bmatrix} n-k-w \\ w-k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-2k \\ w+1-k \end{bmatrix}^{-\frac{1}{2}} \right\}}{[i-u][i-u-1] \cdots [1]} \\ &= q^{\frac{k(i-u)}{2}} \begin{bmatrix} n-k-u \\ i-u \end{bmatrix} \begin{bmatrix} n-2k \\ u-k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-2k \\ i-k \end{bmatrix}^{-\frac{1}{2}}. \end{aligned}$$

Similarly, if we write  $\Phi(M_{u,j}^u) = (B_0^u, \dots, B_m^u)$ , then we have

$$B_k^u = \begin{cases} q^{\frac{k(j-u)}{2}} \begin{bmatrix} n-k-u \\ j-u \end{bmatrix} \begin{bmatrix} n-2k \\ u-k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-2k \\ j-k \end{bmatrix}^{-\frac{1}{2}} E_{u,j,k} & \text{if } k \leq u \leq n-k \\ 0 & \text{otherwise} \end{cases}$$

So from (9) we have  $N_k = \sum_{u=0}^n (-1)^{u-t} q^{\binom{u-t}{2}} \begin{bmatrix} u \\ t \end{bmatrix} A_k^u B_k^u = \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \begin{bmatrix} u \\ t \end{bmatrix} A_k^u B_k^u$ . Thus

$$\begin{aligned} N_k(i, j) &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \begin{bmatrix} u \\ t \end{bmatrix} \left\{ \sum_{l=k}^{n-k} A_k^u(i, l) B_k^u(l, j) \right\} \\ &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \begin{bmatrix} u \\ t \end{bmatrix} A_k^u(i, u) B_k^u(u, j) \\ &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \begin{bmatrix} u \\ t \end{bmatrix} q^{\frac{k(i-u)}{2}} \begin{bmatrix} n-k-u \\ i-u \end{bmatrix} \begin{bmatrix} n-2k \\ u-k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-2k \\ i-k \end{bmatrix}^{-\frac{1}{2}} \\ &\quad \times q^{\frac{k(j-u)}{2}} \begin{bmatrix} n-k-u \\ j-u \end{bmatrix} \begin{bmatrix} n-2k \\ u-k \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-2k \\ j-k \end{bmatrix}^{-\frac{1}{2}} \\ &= q^{\frac{k(i+j)}{2}} \begin{bmatrix} n-2k \\ i-k \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} n-2k \\ j-k \end{bmatrix}^{-\frac{1}{2}} \left\{ \sum_{u=0}^n (-1)^{u-t} q^{\binom{u-t}{2} - ku} \begin{bmatrix} u \\ t \end{bmatrix} \begin{bmatrix} n-k-u \\ i-u \end{bmatrix} \right. \\ &\quad \times \left. \begin{bmatrix} n-k-u \\ j-u \end{bmatrix} \begin{bmatrix} n-2k \\ u-k \end{bmatrix} \right\}, \end{aligned}$$

completing the proof.  $\square$

**Remark** Let  $0 \leq i \leq n/2$ ,  $0 \leq k, t \leq i$  and write  $\Phi(M_{i,i}^t) = (N'_0, \dots, N'_m)$ . Substituting  $j = i$  in Theorem 4.1 and noting that  $\begin{bmatrix} \mathbf{n}-2\mathbf{k} \\ \mathbf{i}-\mathbf{k} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{n}-2\mathbf{k} \\ \mathbf{u}-\mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{n}-\mathbf{k}-\mathbf{u} \\ \mathbf{i}-\mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{i}-\mathbf{k} \\ \mathbf{i}-\mathbf{u} \end{bmatrix}$  we see that  $N'_k = \tau_{i,t,k} E_{i,i,k}$  where

$$\tau_{i,t,k} = \sum_{u=0}^n (-1)^{u-t} q^{\binom{u-t}{2} + k(i-u)} \begin{bmatrix} \mathbf{u} \\ \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{n}-\mathbf{k}-\mathbf{u} \\ \mathbf{i}-\mathbf{u} \end{bmatrix} \begin{bmatrix} \mathbf{i}-\mathbf{k} \\ \mathbf{i}-\mathbf{u} \end{bmatrix}.$$

The  $\tau_{i,t,k}$  are the eigenvalues of the  $q$ -Johnson scheme of  $i$ -dimensional subspaces (see [BI]).

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## References

- [A] M. Aigner, *A course in Enumeration*, Springer-Verlag, Berlin Heidelberg, 2007.
- [BI] E. Bannai, and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, London, 1984.
- [B] L. Butler, *A unimodality result in the enumeration of subgroups of a finite abelian group*, Proceedings of American Mathematical Society, 101 (no. 4): 771-775 (1987).
- [S] A. Schrijver, *New code upper bounds from the Terwilliger algebra and semidefinite programming*, IEEE Transactions on Information Theory, 51: 2859-2866 (2005).
- [S1] M. K. Srinivasan, *Symmetric chains, Gelfand-Tsetlin chains, and the Terwilliger algebra of the binary Hamming scheme*, Journal of Algebraic Combinatorics, 34: 301-322 (2011).
- [S2] M. K. Srinivasan, *Counting spanning trees of the hypercube and its  $q$ -analogs by explicit block diagonalization*, arXiv: 1104.1481.